

# ON THE SIZE OF CERTAIN SUBSETS OF INVARIANT BANACH SEQUENCE SPACES

TONY NOGUEIRA AND DANIEL PELLEGRINO

**ABSTRACT.** The essence of the notion of lineability and spaceability is to find linear structures in somewhat chaotic environments. The existing methods, in general, use *ad hoc* arguments and few general techniques are known. Motivated by the search of general methods, in this paper we formally extend recent results of G. Botelho and V.V. Fávaro on invariant sequence spaces to a more general setting. Our main results show that some subsets of invariant sequence spaces contain, up to the null vector, a closed infinite-dimensional subspace.

## 1. INTRODUCTION AND BACKGROUND

The notion of invariant sequence spaces, as we investigate in this note, was introduced in [8] although it seems to have its roots in [2, 7]. Our main results are formal extensions of recent results of G. Botelho and V.V. Fávaro [9]. We show, among other results, that some special invariant sequence spaces used in [9] can be replaced by more general invariant sequence spaces. Let us first recall the notion of invariant sequence space.

**Definition 1.1.** ([8]) Let  $X \neq \{0\}$  be a Banach space.

- (a) Given  $x \in X^{\mathbb{N}}$ ,  $x^0$  is defined as: if  $x$  has only finitely many non-zero coordinates, then  $x^0 = 0$ ; otherwise,  $x^0 = (x_j)_{j=1}^{\infty}$  where  $x_j$  is the  $j$ -th non-zero coordinate of  $x$ .
- (b) An invariant sequence space over  $X$  is an infinite-dimensional Banach or quasi-Banach space  $E$  of  $X$ -valued sequences enjoying the following conditions:
  - (b1) For  $x \in X^{\mathbb{N}}$  such that  $x^0 \neq 0$ ,  $x \in E$  if and only if  $x^0 \in E$ , and  $\|x\| \leq K\|x^0\|$  for some constant  $K$  depending only on  $E$ .
  - (b2)  $\|x_j\| \leq \|x\|$  for every  $x = (x_j)_{j=1}^{\infty} \in E$  and every  $j \in \mathbb{N}$ .

**Example 1.2.** As mentioned in [8], usual sequence spaces are invariant sequence spaces. For instance

- (a) For every  $0 < p \leq \infty$ , the spaces

$$\begin{aligned} \ell_p(X) &= \left\{ (x_j)_{j=1}^{\infty} \in X^{\mathbb{N}} : \|(x_j)_{j=1}^{\infty}\| := \left( \sum_{j=1}^{\infty} \|x_j\|^p \right)^{\frac{1}{p}} < \infty \right\}, \\ \ell_p^w(X) &= \left\{ (x_j)_{j=1}^{\infty} \in X^{\mathbb{N}} : \|(x_j)_{j=1}^{\infty}\|_{w,p} := \sup_{\|\varphi\| \leq 1} \left( \sum_{j=1}^{\infty} |\varphi(x_j)|^p \right)^{\frac{1}{p}} < \infty, \varphi \in X' \right\}, \\ \ell_p^u(X) &= \left\{ (x_j)_{j=1}^{\infty} \in \ell_p^w(X) : \lim_{n \rightarrow \infty} \|(x_j)_{j=n}^{\infty}\|_{w,p} = 0 \right\}, \\ c_0(X) &= \left\{ (x_j)_{j=1}^{\infty} \in X^{\mathbb{N}} : \lim_{j \rightarrow \infty} x_j = 0 \right\}, \\ c(X) &= \left\{ (x_j)_{j=1}^{\infty} \in X^{\mathbb{N}} : \lim_{j \rightarrow \infty} x_j \text{ exists} \right\} \end{aligned}$$

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are invariant sequence spaces over  $X$ . Above and henceforth,  $X'$  denotes the topological dual of  $X$ , and in  $c_0(X)$  and  $c(X)$  we consider the sup norm. When  $p = \infty$ , the sums are replaced by a supremum and  $\ell_\infty^w(X) := \ell_\infty(X)$ .

(b) For  $0 < p, q < \infty$ , the Lorentz space  $\ell_{p,q}$  is an invariant sequence space (over  $\mathbb{K}$ ). For more details and examples we refer to [8].

The spirit of the concept of lineability and spaceability is to look for linear structures in nonlinear settings. There are few general methods (see, for instance, [4, 5]) to prove lineability and spaceability and, in general, particular problems need *ad hoc* arguments. For more details on the subject we refer to [1, 3, 6] and the references therein.

The following definition is a natural extension of [9, Definition 2.2]:

**Definition 1.3.** Let  $X$  and  $Y$  be Banach spaces,  $\Gamma$  be an arbitrary set and  $E$  be an invariant sequence space over  $X$ . If  $E_l$ , for all  $l \in \Gamma$ , are invariant sequence spaces over  $Y$  and  $f : X \rightarrow Y$  is any map, we define the set

$$G(E, f, (E_l)_{l \in \Gamma}) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{l \in \Gamma} E_l \right\}.$$

According to [9, Definition 2.3] a map  $f : X \rightarrow Y$  between normed spaces is said to be:

(a) *Non-contractive* if  $f(0) = 0$  and for every scalar  $\alpha \neq 0$  there is a constant  $K(\alpha) > 0$  such that

$$(1) \quad \|f(\alpha x)\|_Y \geq K(\alpha) \cdot \|f(x)\|_Y$$

for every  $x \in X$ .

(b) *Strongly non-contractive* if  $f(0) = 0$  and for every scalar  $\alpha \neq 0$  there is a constant  $K(\alpha) > 0$  such that

$$(2) \quad |\varphi(f(\alpha x))| \geq K(\alpha) \cdot |\varphi(f(x))|$$

for all  $x \in X$  and  $\varphi \in Y'$ .

The following result was recently proved by Botelho and Fávoro (see [9, Theorem 2.5]):

**Theorem 1.4.** ([9]) *Let  $X$  and  $Y$  be Banach spaces,  $E$  be an invariant sequence space over  $X$ ,  $f : X \rightarrow Y$  be a function and  $\Gamma \subseteq (0, \infty]$ .*

(a) *If  $f$  is non-contractive, then*

$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\},$$

$$C(E, f, 0) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin c_0(Y) \right\}$$

*are either empty or spaceable in  $E$  (i.e., the union of the set with  $\{0\}$  contains a closed infinite-dimensional subspace of  $E$ ).*

(b) *If  $f$  is strongly non-contractive, then*

$$C^w(E, f, \Gamma) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{q \in \Gamma} \ell_q^w(Y) \right\}$$

*is either empty or spaceable in  $E$ .*

In the present paper we formally extend Theorem 1.4 to a more general setting.

## 2. SPACEABILITY AND STRONGLY INVARIANT SEQUENCE SPACES

From now on an invariant sequence space  $E$  over a Banach space  $X$  will be called strongly invariant sequence space when:

(a)  $c_{00}(X) \subset E$ ;

(b)  $(x_j)_{j=1}^\infty \in E$  if, and only if, all subsequences of  $(x_j)_{j=1}^\infty$  also belong to  $E$ .

**Example 2.1.** If  $X$  is a Banach space, then  $\ell_q(X)$ ,  $\ell_q^u(X)$ ,  $\ell_q^w(X)$ ,  $c(X)$ ,  $c_0(X)$  are strongly invariant sequence spaces.

The notion of strongly invariant sequence space is quite natural, but the following example shows that there exist invariant sequence spaces which are not strongly invariant sequence spaces:

**Example 2.2.** The Banach space  $E = \{(x_j)_{j=1}^\infty \in \ell_\infty : x_{2n-1} = x_{2n} \text{ for all positive integers } n\}$ , with the supremum norm, is an invariant sequence space but is not an strongly invariant sequence space.

The following definition shall be used in the statement of our first main result and in Section 3.

**Definition 2.3.** Let  $X, Y$  be Banach spaces and  $E$  be an invariant sequence space over  $Y$ . A map  $f : X \rightarrow Y$  such that  $f(0) = 0$  is said to be compatible with  $E$  if for any sequence  $(x_j)_{j=1}^\infty$  of elements of  $X$ , we have

$$(f(x_j))_{j=1}^\infty \notin E \Rightarrow (f(ax_j))_{j=1}^\infty \notin E$$

regardless of the scalar  $a \neq 0$ .

**Example 2.4.** Any non-contractive mapping  $f : X \rightarrow Y$  is compatible with  $\ell_q(Y)$  and  $c_0(Y)$ .

Now we state and prove one of the the main results of this paper. We show that [9, Theorem 2.5 (a)] can be formally generalized to a more general setting. The proof is an abstraction of the proof of Theorem 1.4(a):

**Theorem 2.5.** Let  $X$  and  $Y$  be Banach spaces,  $\Gamma$  be an arbitrary set,  $E$  be an invariant sequence space over  $X$  and  $E_l$  be strongly invariant sequence spaces over  $Y$  for all  $l$  in  $\Gamma$ . If  $f : X \rightarrow Y$  is compatible with  $E_l$  for all  $l \in \Gamma$ , then  $G(E, f, (E_l)_{l \in \Gamma})$  is either empty or spaceable.

*Proof.* Assume that  $G(E, f, (E_l)_{l \in \Gamma})$  is non-void and consider  $x = (x_j)_{j=1}^\infty \in G(E, f, (E_l)_{l \in \Gamma})$ . Note that there are infinitely many indexes  $j$  such that  $x_j \neq 0$  because  $f(0) = 0$  and  $c_{00}(Y) \subset E_l$  for all  $l$ . We thus conclude that  $x^0 \neq 0$ . We shall first show that

$$x^0 \in G(E, f, (E_l)_{l \in \Gamma}).$$

Let  $U := \bigcup_{l \in \Gamma} E_l$ . We know that  $(f(x_j))_{j=1}^\infty \notin U$ , and thus  $[(f(x_j))_{j=1}^\infty]^0 \notin U$ , since  $E_l$ , for each  $l$ , is an invariant sequence space. Denote  $x^0 = (x_{j_k})_{k=1}^\infty$ , where  $x_{j_k}$  is the  $k$ -th non null coordinate of  $x$ . Then, we shall show that  $(f(x_{j_k}))_{k=1}^\infty \notin U$ . Since  $f(0) = 0$ , it follows that

$$[(f(x_{j_k}))_{k=1}^\infty]^0 = [(f(x_j))_{j=1}^\infty]^0 \notin U.$$

Hence,  $(f(x_{j_k}))_{k=1}^\infty \notin U$  and thus  $x^0 \in G(E, f, (E_l)_{l \in \Gamma})$ . As usual, let us split  $\mathbb{N}$  as a countable union of pairwise disjoint subsets  $(\mathbb{N}_i)_{i=1}^\infty$  of  $\mathbb{N}$  and we denote  $\mathbb{N}_i = \{i_1 < i_2 < \dots\}$ . Consider

$$y_i = \sum_{k=1}^\infty x_{j_k} \otimes e_{i_k} \in X^{\mathbb{N}}.$$

Observe that  $y_i^0 = x^0$ ; we thus have  $0 \neq y_i^0 \in E$  for all  $i$ . Since  $E$  is an invariant sequence space, it follows that  $y_i \in E$  for all  $i \in \mathbb{N}$ . Note also that the set  $\{y_1, y_2, \dots\}$  is linearly independent. Moreover,  $y_i \in G(E, f, (E_l)_{l \in \Gamma})$ . In fact, if  $y_i = (y_m^i)_{m=1}^\infty$ , then

$$[(f(y_m^i))_{m=1}^\infty]^0 = [(f(x_j))_{j=1}^\infty]^0 \notin E_l$$

for each  $i \in \mathbb{N}$  and  $l \in \Gamma$ . Let  $K$  be the constant from Definition 1.1 (b1) and consider  $\tilde{s} = 1$  if  $E$  is a Banach space and  $\tilde{s} = s$  if  $E$  is an  $s$ -Banach space,  $0 < s < 1$ . For  $(a_i)_{i=1}^\infty \in \ell_{\tilde{s}}$ ,

$$\begin{aligned} \sum_{i=1}^\infty \|a_i y_i\|_E^{\tilde{s}} &= \sum_{i=1}^\infty |a_i|^{\tilde{s}} \cdot \|y_i\|_E^{\tilde{s}} \leq K^{\tilde{s}} \cdot \sum_{i=1}^\infty |a_i|^{\tilde{s}} \cdot \|y_i^0\|_E^{\tilde{s}} \\ &= K^{\tilde{s}} \cdot \|x^0\|_E^{\tilde{s}} \cdot \sum_{i=1}^\infty |a_i|^{\tilde{s}} = K^{\tilde{s}} \cdot \|x^0\|_E^{\tilde{s}} \cdot \|(a_i)_{i=1}^\infty\|_{\tilde{s}}^{\tilde{s}} < \infty. \end{aligned}$$

Then  $\sum_{i=1}^{\infty} \|a_i y_i\|_E^{\bar{s}} < \infty$  and in any case we conclude that  $\sum_{i=1}^{\infty} a_i y_i$  converges in  $E$ . Thus the operator

$$T: \ell_{\bar{s}} \longrightarrow E, \quad T((a_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} a_i y_i,$$

is well-defined, linear and injective. Let us show that  $\overline{T(\ell_{\bar{s}})}$  belongs to  $G(E, f, (E_l)_{l \in \Gamma})$ .

Let us recall that  $y_i = \sum_{k=1}^{\infty} x_{j_k} \otimes e_{i_k} \in X^{\mathbb{N}}$  where  $x_{j_k}$  is the  $k$ -th non-zero coordinate  $x = (x_j)_{j=1}^{\infty} \in G(E, f, (E_l)_{l \in \Gamma})$ . We shall show that if  $z = (z_n)_{n=1}^{\infty} \in \overline{T(\ell_{\bar{s}})}$  is a non null sequence then  $(f(z_n))_{n=1}^{\infty} \notin \bigcup_{l \in \Gamma} E_l$ . There are sequences  $(a_i^{(k)})_{i=1}^{\infty} \in \ell_{\bar{s}}, k \in \mathbb{N}$ , such that  $z = \lim_{k \rightarrow \infty} T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$  in  $E$ . Note that, for each  $k \in \mathbb{N}$ ,

$$T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} a_i^{(k)} y_i = \sum_{i=1}^{\infty} a_i^{(k)} \cdot \sum_{p=1}^{\infty} x_{j_p} \otimes e_{i_p} = \sum_{i=1}^{\infty} \sum_{p=1}^{\infty} a_i^{(k)} x_{j_p} \otimes e_{i_p}.$$

Since  $z \neq 0$ , let  $r \in \mathbb{N}$  be such that  $z_r \neq 0$ . Since  $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$ , there exist unique  $m, t \in \mathbb{N}$  such that  $e_{m_t} = e_r$ . Thus, for each  $k \in \mathbb{N}$ , the  $r$ -th coordinate of  $T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$  is the vector  $a_m^{(k)} x_{j_t}$ . The condition 1.1(b2) of Definition 1.1 assures that convergence in  $E$  implies coordinatewise convergence, so

$$z_r = \lim_{k \rightarrow \infty} a_m^{(k)} x_{j_t} = \left( \lim_{k \rightarrow \infty} a_m^{(k)} \right) x_{j_t}.$$

It follows that  $\alpha_m := \lim_{k \rightarrow \infty} a_m^{(k)} \neq 0$ . On the one hand we have

$$\alpha_m x_{j_p} = \left( \lim_{k \rightarrow \infty} a_m^{(k)} \right) x_{j_p} = \lim_{k \rightarrow \infty} a_m^{(k)} x_{j_p}$$

for every  $p \in \mathbb{N}$ . On the other hand, for  $p, k \in \mathbb{N}$ , the  $m_p$ -th coordinate of  $T\left(\left(a_i^{(k)}\right)_{i=1}^{\infty}\right)$  is  $a_m^{(k)} x_{j_p}$ . So, coordinatewise convergence gives  $\lim_{k \rightarrow \infty} a_m^{(k)} x_{j_p} = z_{m_p}$ . It follows that  $z_{m_p} = \alpha_m x_{j_p}$  for every  $p \in \mathbb{N}$ . As  $(f(z_{m_p}))_{p=1}^{\infty} = (f(\alpha_m x_{j_p}))_{p=1}^{\infty}$  and  $(f(x_{j_p}))_{p=1}^{\infty} \notin E_l$ , for all  $l \in \Gamma$ , by Definition 2.3, it follows that  $(f(z_{m_p}))_{p=1}^{\infty} \notin E_l$ , for all  $l \in \Gamma$ . Since  $(f(z_{m_p}))_{p=1}^{\infty}$  is a subsequence of  $(f(z_n))_{n=1}^{\infty}$  and  $E_l$ , for each  $l \in \Gamma$ , is a strongly invariant sequence space, it follows that

$$(f(z_j))_{j=1}^{\infty} \notin E_l,$$

for all  $l \in \Gamma$ , and it completes the proof that  $z \in G(E, f, (E_l)_{l \in \Gamma})$ .  $\square$

From the previous theorem and Examples 2.1 and 2.4 we have the following corollary that recovers ([9, Theorem 2.5 (a)]):

**Corollary 2.6.** ([9]) *Let  $X$  and  $Y$  be Banach spaces,  $E$  be an invariant sequence space over  $X$ ,  $f: X \longrightarrow Y$  be a function and  $\Gamma \subseteq (0, \infty]$ . If  $f$  is non-contractive, then  $C(E, f, \Gamma)$  and  $C(E, f, 0)$  are either empty or spaceable.*

The next immediate corollary of Theorem 2.5 shows that the [9, Corollaries 2.7, 2.8 and 2.10] and [8, Theorem 1.3] are all particular cases of the following general result:

**Corollary 2.7.** *Let  $X$  and  $Y$  be Banach spaces. Let  $E$  be an invariant sequence space over  $X$  and  $F$  be an strongly invariant sequence space over  $Y$ . If  $f: X \longrightarrow Y$  is compatible with  $F$  and the set*

$$A := \{(x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin F\}$$

*is non empty, then  $A$  is spaceable in  $E$ .*

## 3. THE “WEAK” CASE

In this section we prove an extension of [9, Theorem 2.5(b)], i.e., an extension of Theorem 1.4(b) to more general invariant sequence spaces.

Let  $F$  be an invariant sequence space over  $\mathbb{K}$ . For any Banach space  $Y$  we define

$$F^w(Y) := \{(x_j)_{j=1}^\infty \in Y^\mathbb{N} : (\varphi(x_j))_{j=1}^\infty \in F \text{ for all } \varphi \in Y'\}.$$

It is interesting to remark that if  $F$  is an invariant sequence space over  $\mathbb{K}$ , then

$$(3) \quad \sup_{\|\varphi\| \leq 1} \|(\varphi(x_j))_{j=1}^\infty\|_F < \infty$$

for all  $(x_j)_{j=1}^\infty \in F^w(Y)$ . In fact, if  $(x_j)_{j=1}^\infty \in F^w(Y)$  we consider

$$\begin{aligned} u &: Y' \rightarrow F \\ \varphi &\mapsto (\varphi(x_j))_{j=1}^\infty. \end{aligned}$$

and a general version of the Closed Graph Theorem to topological vector spaces (see [10, page 51]) finish the proof that  $u$  is continuous. Indeed, if

$$\varphi_n \rightarrow \varphi_0 \text{ and } u(\varphi_n) \rightarrow (z_j)_{j=1}^\infty \in F$$

we shall show that  $(z_j)_{j=1}^\infty = u(\varphi_0)$ . Since  $\varphi_n \rightarrow \varphi_0$  we have  $\varphi_n(x_j) \rightarrow \varphi_0(x_j)$  for all  $j$ . On the other hand, since  $u(\varphi_n) \rightarrow (z_j)_{j=1}^\infty$  in  $F$ , i.e.,  $(\varphi_n(x_j))_{j=1}^\infty \rightarrow (z_j)_{j=1}^\infty$  in  $F$ , we also have  $\varphi_n(x_j) \rightarrow z_j$  for all  $j$ . Therefore

$$\varphi_0(x_j) = z_j$$

for all  $j$ , and hence

$$(z_j)_{j=1}^\infty = u(\varphi_0).$$

The next simple lemma highlights some tools to the proof of the main result of this section.

**Lemma 3.1.** *Let  $Y$  be a Banach space. If  $F$  is an strongly invariant sequence space then*

$$(4) \quad x^0 \in F^w(Y) \Leftrightarrow x \in F^w(Y)$$

and

$$(5) \quad c_{00}(Y) \subset F^w(Y)$$

*Proof.* If  $x = (x_j)_{j=1}^\infty$  and  $x^0 = (x_{j_k})_{k=1}^\infty$ , then

$$\begin{aligned} x &\in F^w(Y) \Leftrightarrow (\varphi(x_j))_{j=1}^\infty \in F \text{ for all } \varphi \in Y' \\ &\Leftrightarrow \left( (\varphi(x_j))_{j=1}^\infty \right)^0 \in F \text{ for all } \varphi \in Y' \\ &\Leftrightarrow \left( (\varphi(x_{j_k}))_{k=1}^\infty \right)^0 \in F \text{ for all } \varphi \in Y' \\ &\Leftrightarrow ((\varphi(x_{j_k}))_{k=1}^\infty) \in F \text{ for all } \varphi \in Y' \\ &\Leftrightarrow x^0 = (x_{j_k})_{k=1}^\infty \in F^w(Y). \end{aligned}$$

The proof of (5) is a straightforward consequence of the fact that  $F$  is an strongly invariant sequence space.  $\square$

**Example 3.2.** For  $F = \ell_p, c, c_0$ , the respective  $F^w(Y)$  are the well-known invariant sequence spaces  $\ell_p^w(Y), c^w(Y), c_0^w(Y)$ .

**Definition 3.3.** Let  $X$  and  $Y$  be Banach spaces, and  $F$  be an invariant sequence space over  $\mathbb{K}$ . A map  $f : X \rightarrow Y$  such that  $f(0) = 0$  is strongly compatible with  $F^w(Y)$  if  $\varphi \circ f$  is compatible with  $F$  for all continuous linear functionals  $\varphi : Y \rightarrow \mathbb{K}$ .

**Example 3.4.** Any strongly non-contractive mapping (see (2))  $f : X \rightarrow Y$  is strongly compatible with  $\ell_q^w(Y)$  and  $c_0^w(Y)$ .

**Definition 3.5.** Let  $X$  and  $Y$  be Banach spaces,  $\Gamma$  be an arbitrary set and  $E$  be an invariant sequence space over  $X$ . If  $F_l$ , for all  $l \in \Gamma$ , are invariant sequence spaces over  $\mathbb{K}$ , and  $f : X \rightarrow Y$  is any map, we define the set

$$G^w(E, f, (F_l)_{l \in \Gamma}) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{l \in \Gamma} F_l^w(Y) \right\}.$$

The following theorem is a formal generalization of Theorem 1.4(b). The proof follows the lines of the previous proofs but we present the details for the sake of completeness:

**Theorem 3.6.** *Let  $X$  and  $Y$  be Banach spaces,  $\Gamma$  be an arbitrary set,  $E$  be an invariant sequence space over  $X$  and  $F_l$  be strongly invariant sequence spaces over  $\mathbb{K}$  for all  $l \in \Gamma$ . If  $f : X \rightarrow Y$  is strongly compatible with  $F_l^w(Y)$  for all  $l \in \Gamma$ , then  $G^w(E, f, (F_l)_{l \in \Gamma})$  is either empty or spaceable.*

*Proof.* Assume that  $G^w(E, f, (F_l)_{l \in \Gamma})$  is non-void and consider  $x = (x_j)_{j=1}^\infty \in G^w(E, f, (F_l)_{l \in \Gamma})$ . As in the previous proofs, we shall begin by showing that

$$x^0 \in G^w(E, f, (F_l)_{l \in \Gamma}).$$

Note that  $x_j \neq 0$  for infinitely many  $j$ , because  $f(0) = 0$  and from (5) we have  $c_{00}(Y) \subset F_l^w(Y)$ .

Denote  $U = \bigcup_{l \in \Gamma} F_l^w(Y)$ . We know that for each  $l$  there is a  $\varphi_l$  such that

$$(6) \quad (\varphi_l \circ f(x_j))_{j=1}^\infty \notin F_l,$$

and thus

$$(7) \quad [(\varphi_l \circ f(x_j))_{j=1}^\infty]^0 \notin F_l,$$

for all  $l$ , because  $F_l$ , for each  $l$ , is an invariant sequence space. Denote  $x^0 = (x_{j_k})_{k=1}^\infty$ , where  $x_{j_k}$  is the  $k$ -th non null coordinate of  $x$ . Now, we shall show that

$$(8) \quad (\varphi_l \circ f(x_{j_k}))_{k=1}^\infty \notin F_l$$

for all  $l$ . Suppose that

$$(9) \quad (\varphi_{l_0} \circ f(x_{j_k}))_{k=1}^\infty \in F_{l_0}$$

for some  $l_0$ . From (9), since  $F_{l_0}$  is an invariant sequence space, we would have  $[(\varphi_{l_0} \circ f(x_{j_k}))_{k=1}^\infty]^0 \in F_{l_0}$ . But, since  $\varphi_{l_0} \circ f(0) = 0$ , it would follow from (7) that

$$[(\varphi_{l_0} \circ f(x_{j_k}))_{k=1}^\infty]^0 = [(\varphi_{l_0} \circ f(x_j))_{j=1}^\infty]^0 \notin F_{l_0}.$$

Since  $F_{l_0}$  is an invariant sequence space we would have

$$(\varphi_{l_0} \circ f(x_{j_k}))_{k=1}^\infty \notin F_{l_0}$$

and this contradicts (9). Therefore we have (8), i.e.,

$$(f(x_{j_k}))_{k=1}^\infty \notin F_l^w(Y)$$

for all  $l$ , and thus

$$x^0 \in G^w(E, f, (F_l)_{l \in \Gamma}).$$

Again, let us separate  $\mathbb{N}$  as a countable union of pairwise disjoint subsets  $(\mathbb{N}_i)_{i=1}^\infty$  of  $\mathbb{N}$  and as usual, for all  $i$ , we represent  $\mathbb{N}_i = \{i_1 < i_2 < \dots\}$ . Consider

$$y_i = \sum_{k=1}^\infty x_{j_k} \otimes e_{i_k} \in X^{\mathbb{N}}.$$

Since  $E$  is an invariant sequence space, it follows that  $y_i \in E$  for all  $i \in \mathbb{N}$ . It is plain that  $\{y_1, y_2, \dots\}$  is linearly independent and  $y_i^0 = x^0$ ; we thus have  $0 \neq y_i^0 \in E$  for all  $i$ . Moreover,  $y_i \in G^w(E, f, (F_l)_{l \in \Gamma})$ . In fact, if  $y_i = (y_m^i)_{m=1}^\infty$ , then

$$[(f(y_m^i))_{m=1}^\infty]^0 = [(f(x_j))_{j=1}^\infty]^0 \notin F_l^w(Y)$$

for each  $l$ . Therefore, from (4) of Lemma 3.1, we have

$$(f(y_m^i))_{m=1}^\infty \notin F_l^w(Y)$$

for each  $i \in \mathbb{N}$  and  $l \in \Gamma$  and thus  $y_i \in G^w(E, f, (F_l)_{l \in \Gamma})$ . Let  $\tilde{s} = 1$  if  $E$  is a Banach space and  $\tilde{s} = s$  if  $E$  is an  $s$ -Banach space,  $0 < s < 1$ . Proceeding as in the proof of Theorem 2.5 we know that the operator

$$T: \ell_{\tilde{s}} \longrightarrow E, \quad T((a_i)_{i=1}^\infty) = \sum_{i=1}^\infty a_i y_i,$$

is well-defined and injective. It remains to show that  $\overline{T(\ell_{\tilde{s}})}$  belongs to  $G^w(E, f, (F_l)_{l \in \Gamma})$ . We shall show that if  $z = (z_n)_{n=1}^\infty \in \overline{T(\ell_{\tilde{s}})}$  is a non null sequence then  $(f(z_n))_{n=1}^\infty \notin \bigcup_{l \in \Gamma} F_l^w(Y)$ . Let  $(a_i^{(k)})_{i=1}^\infty \in \ell_{\tilde{s}}$ ,  $k \in \mathbb{N}$ , be such that  $z = \lim_{k \rightarrow \infty} T\left((a_i^{(k)})_{i=1}^\infty\right)$  in  $E$ . Note that, for each  $k \in \mathbb{N}$ ,

$$T\left((a_i^{(k)})_{i=1}^\infty\right) = \sum_{i=1}^\infty a_i^{(k)} y_i = \sum_{i=1}^\infty a_i^{(k)} \cdot \sum_{p=1}^\infty x_{j_p} \otimes e_{i_p} = \sum_{i=1}^\infty \sum_{p=1}^\infty a_i^{(k)} x_{j_p} \otimes e_{i_p}.$$

Fix  $r \in \mathbb{N}$  such that  $z_r \neq 0$ . Since  $\mathbb{N} = \bigcup_{j=1}^\infty \mathbb{N}_j$ , there are (unique)  $m, t \in \mathbb{N}$  such that  $e_{m_t} = e_r$ . Thus, for each  $k \in \mathbb{N}$ , the  $r$ -th coordinate of  $T\left((a_i^{(k)})_{i=1}^\infty\right)$  is the vector  $a_m^{(k)} x_{j_t}$ . From the Definition 1.1(b2) we know that convergence in  $E$  implies coordinatewise convergence, and thus

$$z_r = \lim_{k \rightarrow \infty} a_m^{(k)} x_{j_t} = \left( \lim_{k \rightarrow \infty} a_m^{(k)} \right) x_{j_t}.$$

It follows that  $\alpha_m := \lim_{k \rightarrow \infty} a_m^{(k)} \neq 0$  and

$$\alpha_m x_{j_p} = \left( \lim_{k \rightarrow \infty} a_m^{(k)} \right) x_{j_p} = \lim_{k \rightarrow \infty} a_m^{(k)} x_{j_p}$$

for every  $p \in \mathbb{N}$ . Besides, for  $p, k \in \mathbb{N}$ , the  $m_p$ -th coordinate of  $T\left((a_i^{(k)})_{i=1}^\infty\right)$  is  $a_m^{(k)} x_{j_p}$ . So, coordinatewise convergence gives  $\lim_{k \rightarrow \infty} a_m^{(k)} x_{j_p} = z_{m_p}$  and hence

$$z_{m_p} = \alpha_m x_{j_p}$$

for every  $p \in \mathbb{N}$ . Therefore

$$(10) \quad (\varphi_l \circ f(z_{m_p}))_{p=1}^\infty = (\varphi_l \circ f(\alpha_m x_{j_p}))_{p=1}^\infty$$

Since  $f$  is strongly compatible with  $F_l^w(Y)$  for all  $l$ , we conclude that  $\varphi_l \circ f$  is compatible with  $F_l$  and thus, from (8), it follows that

$$(11) \quad (\varphi_l \circ f(\alpha_m x_{j_p}))_{p=1}^\infty \notin F_l.$$

Thus, from (10) and (11) we have

$$(12) \quad (\varphi_l \circ f(z_{m_p}))_{p=1}^\infty \notin F_l,$$

for all  $l$ . Therefore since  $(\varphi_l \circ f(z_{m_p}))_{p=1}^\infty$  is a subsequence of  $(\varphi_l \circ f(z_n))_{n=1}^\infty$  and  $F_l$  is a strongly invariant sequence space, it follows that

$$(\varphi_l \circ f(z_j))_{j=1}^\infty \notin F_l,$$

for all  $l \in \Gamma$ , and we finally conclude that  $z \in G^w(E, f, (F_l)_{l \in \Gamma})$ .  $\square$

Let  $X$  and  $Y$  be Banach spaces,  $E$  be an invariant sequence space over  $X$  and  $F_l = \ell_l$ , with  $l \in \Gamma \subset (0, \infty]$ . If  $f: X \rightarrow Y$  is strongly non-contractive then  $f$  is strongly compatible with  $\ell_l^w(Y)$  and from the previous theorem we conclude that  $G^w(E, f, (F_l)_{l \in \Gamma})$  is either empty or spaceable. Since

$$G^w(E, f, (F_l)_{l \in \Gamma}) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{l \in \Gamma} \ell_l^w(Y) \right\}$$

we recover Theorem 1.4(b).

**Remark 3.7.** It is interesting to note that in all the results presented in this note (and in the respective versions from [2, 9]) the lineability/spaceability results satisfy a somewhat slightly stronger condition, in the following sense: given any point  $x$  of the set  $G(E, f, (E_l)_{l \in \Gamma})$  it is proved here that there is a closed infinite dimensional vector space  $V$  such that “essentially”  $x \in V \subset G(E, f, (E_l)_{l \in \Gamma}) \cup \{0\}$ . This leads to the following extension of the notion of lineability that may be interesting to investigate in different contexts: a subset  $A$  of a vector space  $W$  is *pointwise  $\lambda$ -lineable* if for any  $x \in A$  there is a  $\lambda$ -dimensional vector space  $V$  such that  $x \in V \subset A \cup \{0\} \subset W$ . The same definition can be adapted to the notion of spaceability. It is not difficult to verify that in general these concepts are strictly stronger than just lineability/spaceability. For instance, let  $W = \ell_2$  and

$$A = (\text{span}\{e_1\}) \cup (\text{span}\{e_2, e_3\}) \cup (\text{span}\{e_4, \dots, e_6\}) \cup \dots$$

It is plain that  $A$  is  $n$ -lineable for all positive integer  $n$ , but  $A$  is not pointwise 2-lineable.

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA PARAÍBA, 58.051-900 - JOÃO PESSOA, BRAZIL.  
E-mail address: tonyklevererson@gmail.com

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA PARAÍBA, 58.051-900 - JOÃO PESSOA, BRAZIL.  
E-mail address: pellegrino@pq.cnpq.br and dpellegrino@gmail.com